Nano-Optics

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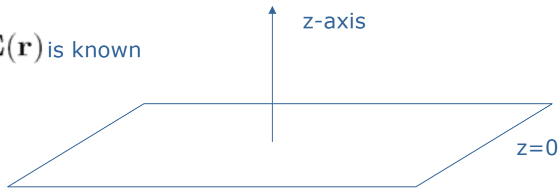
Table of Contents

Angular spectrum representation of optical fields

- mathematical technique to describe optical fields in homogeneous media
- Optical fields are described as a superposition of plane waves and evanescent waves which are physically intuitive solutions of Maxwell's equations.

$\mathbf{E}(\mathbf{r})$ is known



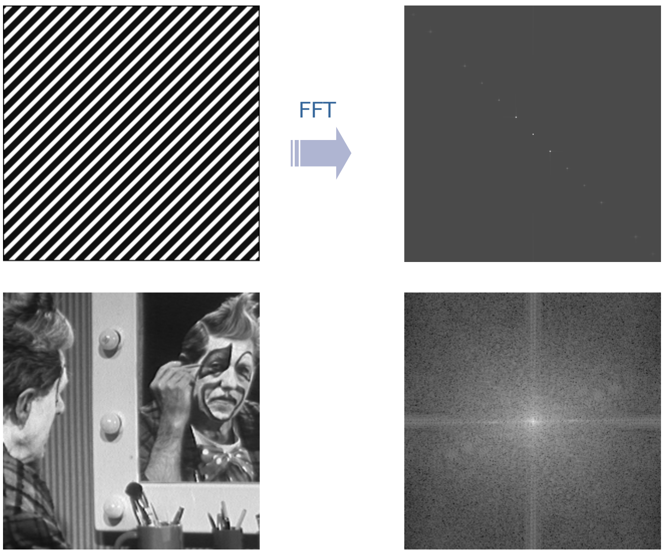
$$\hat{\mathbf{E}}(k_x, k_y; 0) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \mathbf{E}(x, y, 0)\, e^{-i[k_x x + k_y y]}\, dx\, dy$$

two-dimensional Fourier transform of the field

k_x, k_y spatial frequencies

Similar to image processing using Fourier transform: http://homepages.inf.ed.ac.uk/rbf/HIPR2/fourier.htm

Spatial frequencies



http://homepages.inf.ed.ac.uk/rbf/HIPR2/fourier.htm inverse Fourier transform:

$$\mathbf{E}(x,y,0) = \iint_{-\infty}^{\infty} \hat{\mathbf{E}}(k_x,k_y;0)\, e^{i[k_x x + k_y y]}\, dk_x\, dk_y$$

the Fourier integrals hold separately for each vector component

homogeneous, isotropic, linear and source-free medium:

$$(\nabla^2 + k^2)\,\mathbf{E}(\mathbf{r}) = 0 \qquad k = (\omega/c)\,n \qquad n = \sqrt{\mu\varepsilon}$$

$$\hat{\mathbf{E}}(k_x,k_y;z) = \hat{\mathbf{E}}(k_x,k_y;0)\, e^{\pm i k_z z}$$

tells us how to calculate the Fourier coefficients for any z once the coefficioents for z=0 are known!

Impossibility of evanescent waves in free space!

Ende! 26.04.2007

$$k_z \equiv \sqrt{(k^2 - k_x^2 - k_y^2)}$$

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5

$$\mathbf{E}(x,y,z) = \iint_{-\infty}^{\infty} \hat{\mathbf{E}}(k_x,k_y;0)\, e^{i[k_x x + k_y y \pm k_z z]}\, dk_x\, dk_y$$

Types of waves involved:

$$\text{Plane waves}: \quad e^{i[k_x x + k_y y]}\, e^{\pm i|k_z|z}, \qquad k_x^2 + k_y^2 \le k^2$$

$$\text{Evanescent waves}: \quad e^{i[k_x x + k_y y]}\, e^{-|k_z||z|}, \qquad k_x^2 + k_y^2 > k^2$$

Angular spectrum representation:
superposition of *plane waves*
and *evanescent waves!*

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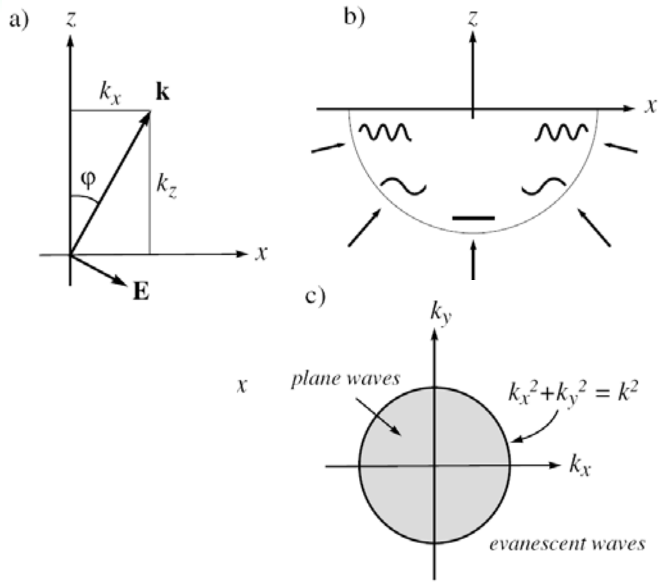
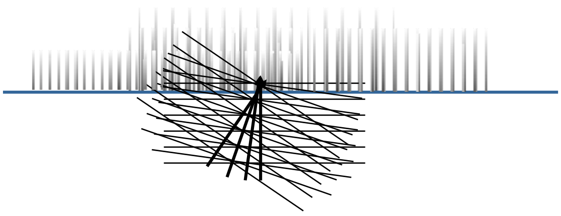
6

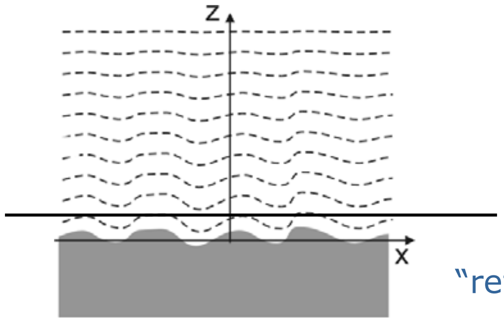


Visualization: Angular spectrum



Example: optical imaging

Field above an illuminated sample



z=0

"reference plane"

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Example: optical imaging

2D Fourier analysis:

$$\mathbf{E}(x,y,z) = \iint_{-\infty}^{\infty} \hat{\mathbf{E}}(k_x,k_y;0)\, e^{i\,[k_x x + k_y y \pm k_z z]}\, dk_x\, dk_y$$

$$k_z \equiv \sqrt{(k^2 - k_x^2 - k_y^2)}$$

$$\text{Plane waves:} \quad e^{i\,[k_x x + k_y y]}\, e^{\pm i |k_z| z}, \qquad k_x^2 + k_y^2 \le k^2$$

larger features

$k^{-1}=\lambda/2\pi\sim 100$nm

$$\text{Evanescent waves:} \quad e^{i\,[k_x x + k_y y]}\, e^{-|k_z||z|}, \qquad k_x^2 + k_y^2 > k^2$$

smaller (subwavelength) features

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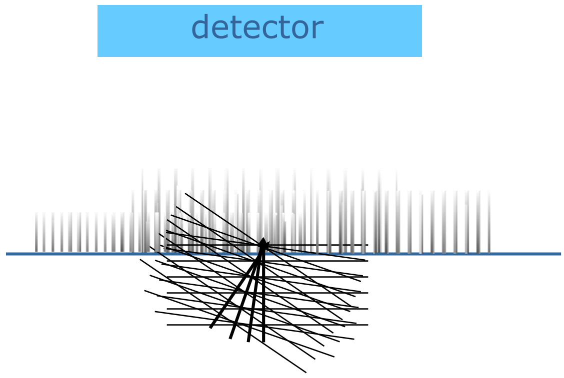
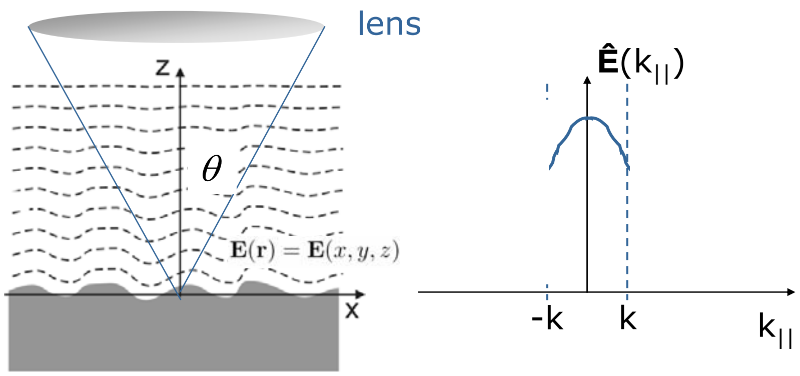
Far-field detection



Diffraction limit "again"



$$\Delta x \approx \frac{1}{k} = \frac{\lambda}{2\pi n}$$

Focusing of light

is of great importance in

- Confocal microscopy
- data storage (CD, DVD, …)
- optical tweezers
- single-molecule spectroscopy
- material processing
- …..

Focusing provides the best light confinement possible with far-field optical methods!

Paraxial approximation

$$k_z = k\sqrt{1-(k_x^2+k_y^2)/k^2} \approx k - \frac{(k_x^2+k_y^2)}{2k}$$

Gaussian beam

$$\mathbf{E}(x', y', 0) \;=\; \mathbf{E}_o\, \mathrm{e}^{-\frac{x'^2+y'^2}{w_o^2}}$$

field distribution in the beam waist, z=0

2d Gaussian in (x,y), z=0

ω_o beam waist radius

Spatial Fourier spectrum at z=0

$$\hat{\mathbf{E}}(k_x, k_y; 0) \;=\; \frac{1}{4\pi^2} \iint\limits_{-\infty}^{\infty} \mathbf{E}_o\, \mathrm{e}^{-\frac{x'^2+y'^2}{w_o^2}}\, \mathrm{e}^{-i\,[k_x x' + k_y y']}\, dx'\, dy'$$

$$= \; \mathbf{E}_o \frac{w_o^2}{4\pi}\, \mathrm{e}^{-(k_x^2+k_y^2)\frac{w_o^2}{4}}$$

again a Gaussian

$$\int_{-\infty}^{\infty} \exp(-ax^2 \;+\; ibx)\, dx \;=\; \sqrt{\pi/a}\, \exp(-b^2/4a)$$

$$\int_{-\infty}^{\infty} x \exp(-ax^2 \;+\; ibx)\, dx \;=\; ib\sqrt{\pi}\, \exp(-b^2/4a)/(2a^{3/2})$$

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Gaussian beam

Use:

$$\mathbf{E}(x, y, z) \;=\; \iint\limits_{-\infty}^{\infty} \hat{\mathbf{E}}(k_x, k_y; 0)\, \mathrm{e}^{i\,[k_x x \,+\, k_y y \,\pm\, k_z z]}\, dk_x\, dk_y$$

$$\mathbf{E}(x, y, z) \;=\; \mathbf{E}_o \frac{w_o^2}{4\,\pi}\, \mathrm{e}^{i\,k z} \iint\limits_{-\infty}^{\infty} \mathrm{e}^{-(k_x^2+k_y^2)(\frac{w_o^2}{4}+\frac{i\,z}{2\,k})}\, \mathrm{e}^{i\,[k_x x \,+\, k_y y]}\, dk_x\, dk_y$$

Paraxial representation of a Gaussian beam:

$$\mathbf{E}(x, y, z) \;=\; \frac{\mathbf{E}_o\, \mathrm{e}^{i\,k z}}{(1 \,+\, 2\,i\,z/kw_o^2)}\, \mathrm{e}^{-\frac{(x^2+y^2)}{w_o^2}\frac{1}{(1\,+\,2\,i\,z\,/\,k w_o^2)}}$$

use polar coordinates: $\rho^2 = x^2 + y^2$ and $z_o \;=\; \dfrac{k\, w_o^2}{2}$

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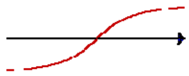
Gaussian beam

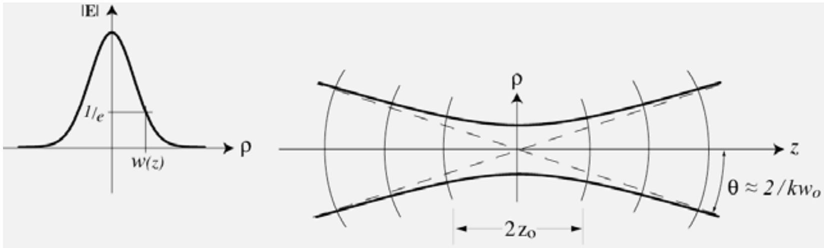
$$\mathbf{E}(\rho, z) = \mathbf{E}_o \frac{w_o}{w(z)}\, e^{-\frac{\rho^2}{w^2(z)}}\, e^{i\,[kz - \eta(z) + k\rho^2/2R(z)]}$$

$$w(z) = w_o\,(1 + z^2/z_o^2)^{1/2} \qquad beam\ waist$$

$$R(z) = z\,(1 + z_o^2/z^2) \qquad wavefront\ radius$$

$$\eta(z) = \arctan z/z_o \qquad phase\ correction$$





Rayleigh range

increase of the beam radius by a factor of $\sqrt{2}$

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17

Gaussian beam

- Gaussian beam in paraxial approximation is no longer a solution of Maxwell´s equations. The error becomes larger the smaller the beam waist radius ω_o is.
- Gaussian beams do not exist! The spectrum allways contains unphysical evanescent components.

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18

Longitudinal fields in the focal region

x-polarized Gaussian beam:

$$\nabla \cdot \mathbf{E} = 0 \quad \Rightarrow \quad E_z = -\int \left[\frac{\partial}{\partial x} E_x \right] dz$$

$$E_z(x,y,0) = -i \frac{2x}{kw_o^2} E_x(x,y,0)$$

in the focal plane

- stronger for smaller beam waist
- 90^o phase shift
- zero on the optical axis, two lobes along the direction of polarization

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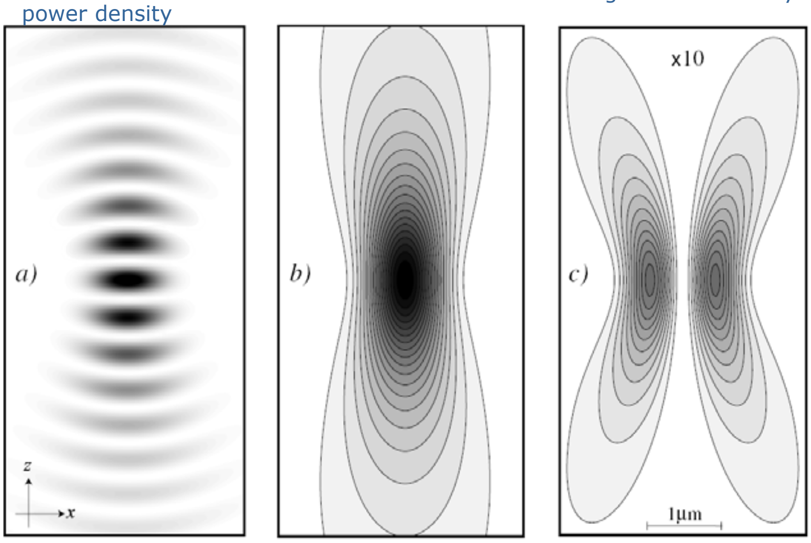
Longitudinal fields in the focal region

Time-dependent power density | Total intensity | Longitudinal intensity



$$\lambda = 800\,nm \qquad \theta = 28.65^\circ$$

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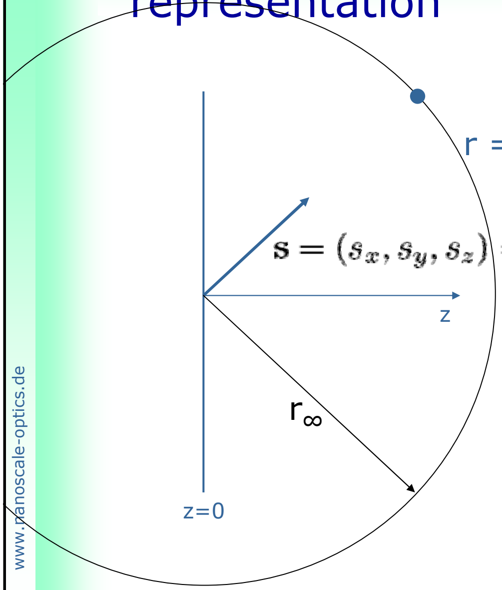
Far fields in the angular spectrum representation



$$\mathbf{s} = (s_x, s_y, s_z) = (\frac{x}{r}, \frac{y}{r}, \frac{z}{r})$$ directional cosine

$$r = (x^2 + y^2 + z^2)^{1/2}$$

each point on the sphere r = ∞ corresponds to one direction described by **s**

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21

Far fields in the angular spectrum representation

$$\mathbf{E}(x, y, z) = \iint_{-\infty}^{\infty} \hat{\mathbf{E}}(k_x, k_y; 0)\, e^{i\,[k_x x + k_y y \pm k_z z]}\, dk_x\, dk_y$$

$$\mathbf{s} = (s_x, s_y, s_z) = (\frac{x}{r}, \frac{y}{r}, \frac{z}{r})$$

$k\, r\, k_x/x\, s_x = k_x x$

$$\mathbf{E}_\infty(s_x, s_y, s_z) = \lim_{kr\to\infty} \iint_{(k_x^2+k_y^2)\le k^2} \hat{\mathbf{E}}(k_x, k_y; 0)\, e^{i\,kr\,[\frac{k_x}{k} s_x + \frac{k_y}{k} s_y \pm \frac{k_z}{k} s_z]}\, dk_x\, dk_y$$

method of stationary phase:

$$\mathbf{E}_\infty(s_x, s_y, s_z) = -2\pi i k\, s_z\, \hat{\mathbf{E}}(ks_x, ks_y; 0)\, \frac{e^{ikr}}{r}$$

the farfields are entirely defined by the Fourier spectrum of the fields

see e.g. L. Mandel and E. Wolf, Optical Coherence and Quantum Optics. New York: Cambridge University Press (1995).

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22

Far fields in the angular spectrum representation

$$\mathbf{s} = (s_x, s_y, s_z) = \left(\frac{k_x}{k}, \frac{k_y}{k}, \frac{k_z}{k}\right)$$

$$\hat{\mathbf{E}}(k_x, k_y; 0) = \frac{i r\, e^{-ikr}}{2\pi k}\, \mathbf{E}_\infty(k_x, k_y)$$

Fourier spectrum in terms of the farfield

$$\mathbf{E}(x,y,z) = \frac{i r\, e^{-ikr}}{2\pi} \iint_{(k_x^2+k_y^2)\le k^2} \mathbf{E}_\infty(k_x,k_y)\, e^{i[k_x x + k_y y \pm k_z z]}\, \frac{1}{k_z}\, dk_x\, dk_y$$

As long as evanescent fields are not part of the system the field $\boldsymbol{E}$ and its far field $\boldsymbol{E}_\infty$ form essentially a Fourier transform pair. The only deviation is given by the factor $1/k_z$. In the approximation $k_z \approx k$, the two fields form a perfect Fourier transform pair. Limit of Fourier optics!

Important result. It links the near-fields of an optical problem with the corresponding far fields. While in the near-field a rigorous description of fields is necessary, the far fields are well approximated by the laws of geometrical optics

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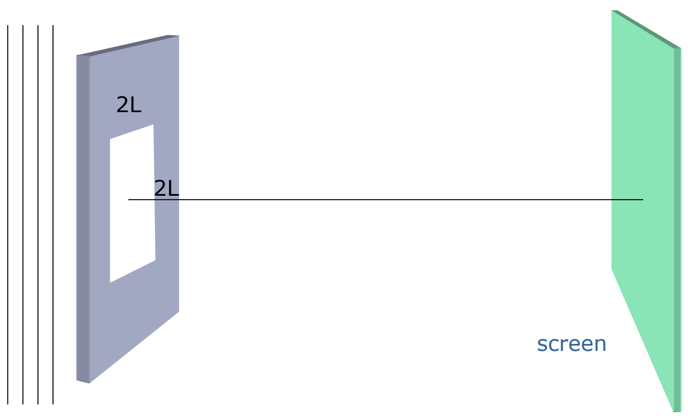
Diffraction @ rectangular aperture



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$$\hat{\mathbf{E}}(k_x, k_y; 0) = \frac{\mathbf{E}_o}{4\pi^2} \int_{-L_y}^{+L_y} \int_{-L_x}^{+L_x} \mathrm{e}^{-i\,[k_x x' + k_y y']}\, dx'\, dy'$$

$$= \mathbf{E}_o \frac{L_x L_y}{\pi^2} \frac{\sin(k_x L_x)}{k_x L_x} \frac{\sin(k_y L_y)}{k_y L_y}$$

Far field:

$$\mathbf{E}_\infty(s_x, s_y, s_z) = -ik\, s_z\, \mathbf{E}_o \frac{2L_x L_y}{\pi} \frac{\sin(ks_x L_x)}{ks_x L_x} \frac{\sin(ks_y L_y)}{ks_y L_y} \frac{\mathrm{e}^{ikr}}{r}$$

for $k_z \approx k$: Fraunhofer diffraction!

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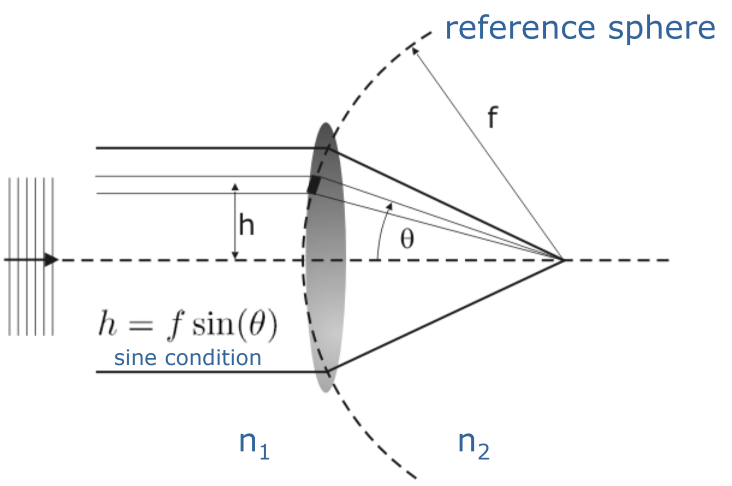
Aplanatic focusing



$$|\mathbf{E}_2| = |\mathbf{E}_1| \sqrt{\frac{n_1}{n_2}} \cos^{1/2}\theta$$ Intensity law

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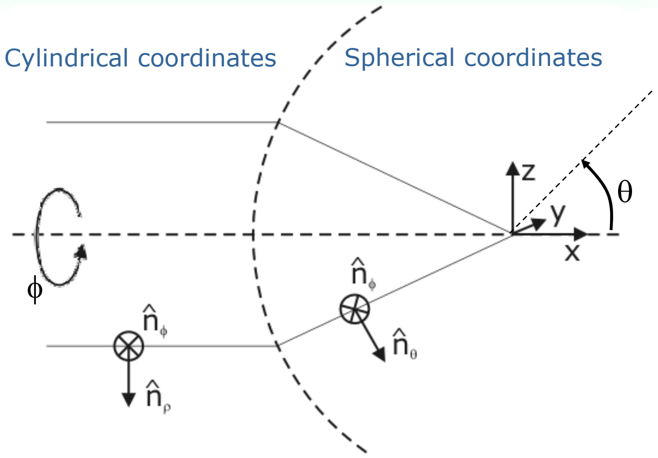
Coordinate systems



$$\mathbf{E}_{inc}^{(s)} = [\mathbf{E}_{inc} \cdot \mathbf{n}_\phi]\ \mathbf{n}_\phi$$

$$\mathbf{E}_{inc}^{(p)} = [\mathbf{E}_{inc} \cdot \mathbf{n}_\rho]\ \mathbf{n}_\rho$$

total refracted electric field

$$\mathbf{E}_\infty = \left[t^s\,[\mathbf{E}_{inc} \cdot \mathbf{n}_\phi]\ \mathbf{n}_\phi \ + \ t^p\,[\mathbf{E}_{inc} \cdot \mathbf{n}_\rho]\ \mathbf{n}_\theta\right] \sqrt{\frac{n_1}{n_2}}\,(\cos\theta)^{1/2}$$

$$\begin{aligned}
\mathbf{n}_\rho &= \cos\phi\,\mathbf{n}_x + \sin\phi\,\mathbf{n}_y\,, \\
\mathbf{n}_\phi &= -\sin\phi\,\mathbf{n}_x + \cos\phi\,\mathbf{n}_y\,, \\
\mathbf{n}_\theta &= \cos\theta\,\cos\phi\,\mathbf{n}_x + \cos\theta\,\sin\phi\,\mathbf{n}_y - \sin\theta\,\mathbf{n}_z
\end{aligned}$$

$$\begin{aligned}
\mathbf{E}_\infty(\theta,\phi) = \ & t^s(\theta)\left[\mathbf{E}_{inc}(\theta,\phi)\cdot\begin{pmatrix}-\sin\phi\\ \cos\phi\\ 0\end{pmatrix}\right]\begin{pmatrix}-\sin\phi\\ \cos\phi\\ 0\end{pmatrix}\sqrt{\frac{n_1}{n_2}}\,(\cos\theta)^{1/2} \\
+ \ & t^p(\theta)\left[\mathbf{E}_{inc}(\theta,\phi)\cdot\begin{pmatrix}\cos\phi\\ \sin\phi\\ 0\end{pmatrix}\right]\begin{pmatrix}\cos\phi\,\cos\theta\\ \sin\phi\,\cos\theta\\ -\sin\theta\end{pmatrix}\sqrt{\frac{n_1}{n_2}}\,(\cos\theta)^{1/2}
\end{aligned}$$

$$\mathbf{E}(x,y,z) = \frac{i r\, e^{-ikr}}{2\pi} \iint\limits_{(k_x^2+k_y^2)\le k^2} \mathbf{E}_\infty(k_x,k_y)\; e^{i[k_x x + k_y y \pm k_z z]}\; \frac{1}{k_z}\, dk_x\, dk_y$$

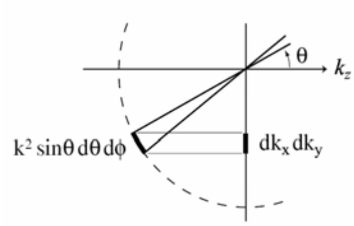
field in the focus determined by the field right after the lens

Express everything in terms of the angles θ and ϕ using:

$$k_x = k\sin\theta\,\cos\phi, \quad k_y = k\sin\theta\,\sin\phi, \quad k_z = k\cos\theta$$

$$x = \rho\cos\varphi, \qquad y = \rho\sin\varphi$$

$$\frac{1}{k_z}\,dk_x\,dk_y = k\,\sin\theta\,d\theta\,d\phi$$



$dk_x dk_y = \cos\theta\,[k^2 \sin\theta\, d\theta\, d\phi]$

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29

$$\mathbf{E}(\rho,\varphi,z) = \frac{ikf\,e^{-ikf}}{2\pi} \int_0^{\theta_{max}}\int_0^{2\pi} \mathbf{E}_\infty(\theta,\phi)\; e^{ikz\cos\theta}\; e^{ik\rho\sin\theta\cos(\phi-\varphi)} \sin\theta\, d\phi\, d\theta$$

$f = r_\infty$ $\qquad NA = n\,\sin\theta_{max}\,, \quad \theta_{max} = [0\,..\,\pi/2]$

Now assume that: $\mathbf{E}_{inc} = E_{inc}\,\mathbf{n}_x$ $\qquad t_\theta^s = t_\theta^p = 1$

$$\mathbf{E}_\infty(\theta,\phi) = E_{inc}(\theta,\phi)\,[\cos\phi\,\mathbf{n}_\theta - \sin\phi\,\mathbf{n}_\phi]\,\sqrt{n_1/n_2}\,(\cos\theta)^{1/2}$$

$$= E_{inc}(\theta,\phi)\,\frac{1}{2}\begin{bmatrix} (1+\cos\theta) - (1-\cos\theta)\cos 2\phi \\ -(1-\cos\theta)\sin 2\phi \\ -2\cos\phi\,\sin\theta \end{bmatrix} \sqrt{\frac{n_1}{n_2}}\,(\cos\theta)^{1/2}$$

To proceed we need to specify the amplitude profile of the incoming beam E_{inc}.

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30

$$(0,0)\ mode:$$

$$E_{inc} = E_o\, \mathrm{e}^{-(x_\infty^2 + y_\infty^2)/w_o^2} \;=\; E_o\, \mathrm{e}^{-f^2 \sin^2\theta / w_o^2}$$

$$f_o \;=\; \frac{w_o}{f\ \sin\theta_{max}}$$ filling factor

$$\Rightarrow\quad f_w(\theta) \;=\; \mathrm{e}^{-\frac{1}{f_o^2}\frac{\sin^2\theta}{\sin^2\theta_{max}}}$$

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31

Mathematical relations:

$$\int_0^{2\pi} \cos n\phi\ \mathrm{e}^{ix\cos(\phi-\varphi)}\, d\phi \;=\; 2\pi(i^n)\, J_n(x)\, \cos n\varphi$$

$$\int_0^{2\pi} \sin n\phi\ \mathrm{e}^{ix\cos(\phi-\varphi)}\, d\phi \;=\; 2\pi(i^n)\, J_n(x)\, \sin n\varphi$$

Abbreviations

$$I_{00} \;=\; \int_0^{\theta_{max}} f_w(\theta)\, (\cos\theta)^{1/2} \sin\theta\, (1+\cos\theta)\, J_0(k\rho\sin\theta)\, \mathrm{e}^{ikz\cos\theta}\, d\theta$$

$$I_{01} \;=\; \int_0^{\theta_{max}} f_w(\theta)\, (\cos\theta)^{1/2} \sin^2\theta\, J_1(k\rho\sin\theta)\, \mathrm{e}^{ikz\cos\theta}\, d\theta$$

$$I_{02} \;=\; \int_0^{\theta_{max}} f_w(\theta)\, (\cos\theta)^{1/2} \sin\theta\, (1-\cos\theta)\, J_2(k\rho\sin\theta)\, \mathrm{e}^{ikz\cos\theta}\, d\theta$$

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32

Focal fields

$$\mathbf{E}(\rho,\varphi,z) = \frac{ikf}{2}\sqrt{\frac{n_1}{n_2}}\,E_o\,\mathrm{e}^{-ikf}\begin{bmatrix} I_{00}+I_{02}\cos 2\varphi \\ I_{02}\sin 2\varphi \\ -2iI_{01}\cos\varphi \end{bmatrix}$$

$$\mathbf{H}(\rho,\varphi,z) = \frac{ikf}{2Z_{\mu\varepsilon}}\sqrt{\frac{n_1}{n_2}}\,E_o\,\mathrm{e}^{-ikf}\begin{bmatrix} I_{02}\sin 2\varphi \\ I_{00}-I_{02}\cos 2\varphi \\ -2iI_{01}\sin\varphi \end{bmatrix}$$

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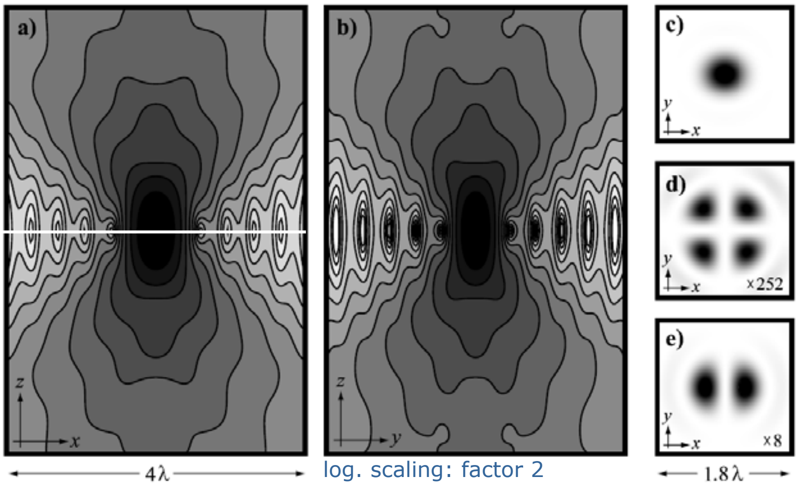
Focal fields

in the focal plane NA=1.4 θ = 0 ... 63°



log. scaling: factor 2 between lines

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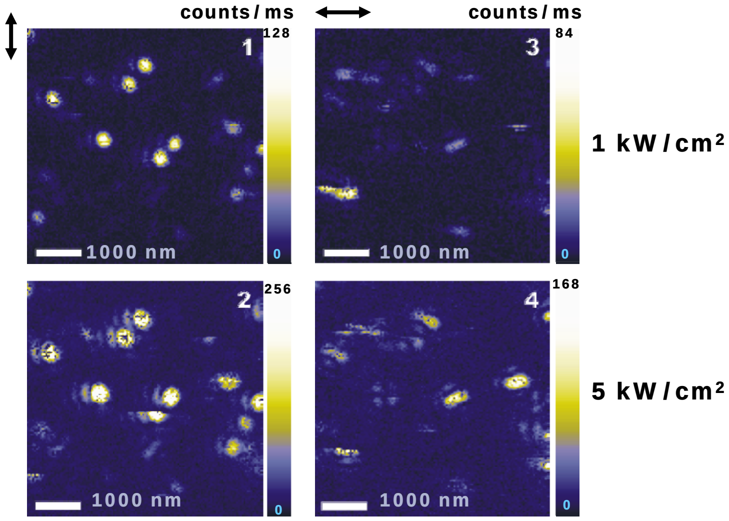
Experimental evidence

Single molecule imaging:



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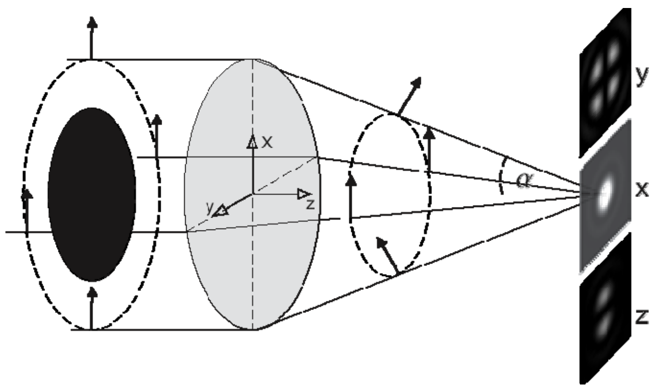
Focal engineering!

Changing the focal field distribution by using different inputs:

Example 1.: Annular illumination



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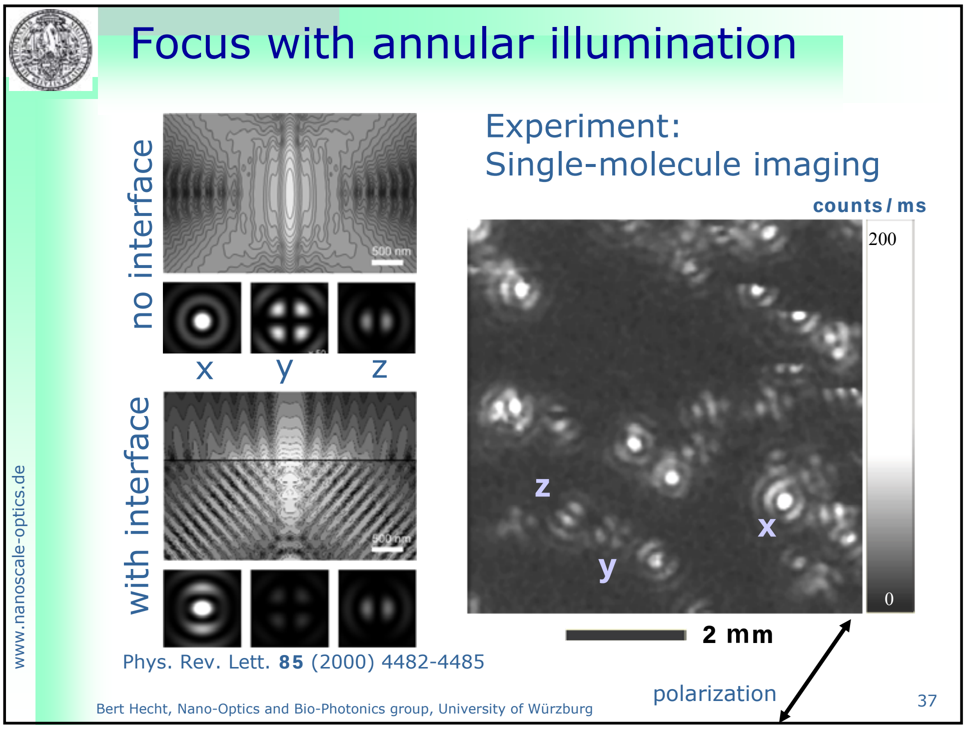
Focus with annular illumination
no interface
500 nm
x
y
z
with interface
500 nm
Experiment:
Single-molecule imaging
counts / ms
200
0
z
y
x
2 mm
polarization
Phys. Rev. Lett. 85 (2000) 4482-4485
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37